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# EXAMPLES OF EXPANDING $C^1$ MAPS HAVING NO $\sigma$-FINITE INVARIANT MEASURE EQUIVALENT TO LEBESGUE

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ABSTRACT

In this paper we construct a $C^1$ expanding circle map with the property that it has no $\sigma$-finite invariant measure equivalent to Lebesgue measure. We extend the construction to interval maps and maps on higher dimensional tori and the Riemann sphere. We also discuss recurrence of Lebesgue measure for the family of tent maps.

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## 1. Introduction

The problem of constructing smooth maps on manifolds with no equivalent $\sigma$-finite invariant measure equivalent to the Riemannian measure has a long history. The map is called **type** *III* (with respect to the measure) in this case. We describe briefly the history of the problem of constructing type *III* maps on manifolds with respect to Lebesgue measure.

Ornstein was the first to construct a type *III* invertible map [24]; it is a continuous interval map. There are by now many examples of type *III* diffeomorphisms, the earliest constructions being $C^\infty$ circle diffeomorphisms [13, 16, 17]. In [12] a noninvertible $C^\infty$ type *III* map of the torus is given. None of these examples is expanding; indeed there are obstructions to finding smooth expanding type *III* maps. This fact is illustrated by a result by Krzyzewski and Szlenk [20], stating that every expanding $C^2$ transformation on a compact manifold carries an absolutely continuous invariant probability measure (acip).

Later, several authors obtained the same result while weakening the $C^2$ assumption for expanding maps, for example [6, 23, 27]. In each of these papers the map was assumed to be $C^1$ and to satisfy an additional condition on the derivative (Hölder, bounded variation etc.). In the $C^1$ expanding case examples have been constructed which admit no finite absolutely continuous measures [7]; however the existence of an infinite $\sigma$-finite invariant measure is not excluded. In addition, many examples of noninvertible $C^\infty$ maps, for instance quadratic maps on the interval, with infinite $\sigma$-finite invariant measures have been found, e.g. [2, 14].

In this paper we construct ergodic type *III* maps of manifolds which are $C^1$ and expanding. Questions on the Lebesgue ergodicity of $C^1$ expanding maps have been addressed by Quas [25] and references therein. We prove the following theorems.

THEOREM 1.1: *There exist maps $f$ for which there is no $\sigma$-finite invariant measure equivalent to Lebesgue. $f$ can be constructed to satisfy one of the following sets of properties.*

- *$f$ is a $C^1$ expanding circle map of degree $d \geq 2$.*
- *$f$ is an interval map topologically conjugate to the full tent map $T_2$, and $f$ is either $C^1$, or $C^1$ and expanding on both branches separately.*
- *$f$ is an interval map topologically conjugate to the tent map with slope $a$, for any $a \in (1, 2]$ such that the critical point has a nowhere dense orbit.*

THEOREM 1.2: *There exists an ergodic type III Borel measure $\nu$ on $\mathbf{C}_\infty$ with respect to the rational map $R(z) = ((z^2+1)^2)/(4z(z^2-1))$.*

We use a construction of Hamachi [9] of a type *III* shift map with a product measure to prove our main result. This gives us in a natural way a type *III* measure for the angle doubling map on the circle which is the basis for our smooth examples.

It is known that for noninvertible maps, equivalent measures can exhibit different recurrence properties. Given a nonsingular ergodic endomorphism $(X, \mathcal{B}, \mu, T)$, $\mu(X) = 1$, the (global) Radon-Nikodým derivative of $T$, denoted $\omega_\mu$, is defined to be the unique $T^{-1}\mathcal{B}$-measurable function satisfying

$$\int_X f \circ T \cdot \omega_\mu d\mu = \int_X f d\mu \quad \text{for all } f \in L^1(X, \mu).$$

The higher derivatives are defined for each $n \in \mathbf{N}$ by $\omega_\mu(n,x) = \prod_{i=0}^{n-1} \omega_\mu(T^i x)$. The measure $\mu$ is said to be **recurrent for** $T$ if

$$\sum_n \omega_\mu(n,x) = \infty \quad \mu\text{-a.e.}$$

In the invertible case, all measures are recurrent. For noninvertible maps, nonrecurrent measures equivalent to recurrent ones are known to exist [5]. In general it is difficult to determine whether a given ergodic measure is recurrent; an invariant measure $\mu$ is always recurrent since $\omega_\mu(x) \equiv 1$ in this case. In a noninvertible system, if the measure is known to be recurrent, all existing invariants for invertible systems (e.g. ratio sets and coboundary Radon–Nikodým derivatives) can be used to detect the existence or absence of an equivalent invariant measure [12]. When the measure is nonrecurrent none of the noninvertible tests are valid. Furthermore, nonrecurrent measures provide obstacles to obtaining a conservative natural extension, even if the original map has an acip [28].

Let $T_a$ be the tent map on $[0,1]$ with constant slope $a$. For $1 < a \leq 2$, $T_a$ is known to admit an acip (e.g. [21]). We prove the following theorem.

THEOREM 1.3: *There are only two values of $a \in (1,2]$, namely $\sqrt{2}$ and 2, for which Lebesgue measure is recurrent.*

The paper is organized as follows: after some definitions and notations, we prove the nonrecurrence of Lebesgue measure for tent maps in Section 3. In Section 4, we briefly discuss Hamachi's construction on the shift space. In the next two sections we apply his construction to obtain our results for circle and interval maps. In the last section we focus on higher dimensional generalizations.

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## 2. Preliminaries

Throughout this paper we will only consider measurable nonsingular transformations $T$ of Lebesgue spaces $X$, where $X$ is usually a manifold, endowed with the $\sigma$-algebra of Borel sets, denoted $\mathcal{B}$, and a Lebesgue measure $\mu$ on $\mathcal{B}$ which is $\sigma$-finite. These assumptions on $T$ mean that for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$, and $\mu(A) = 0$ if and only if $\mu T^{-1}(A) = 0$. By replacing $T$ by an isomorphic copy if necessary, we also assume that $T$ is forward measurable and nonsingular; i.e., for all $A \in \mathcal{B}$, $T(A) \in \mathcal{B}$, and $\mu(A) = 0$ if and only if $\mu T(A) = 0$ [26]. Under these assumptions $T$ will always be surjective as well with respect to $\mu$.

*Definition 2.1:* Let $(X, \mathcal{B}, \mu)$ be a Lebesgue measure space with $\mu$ a finite measure. We say that $T$ is a **bounded-to-one endomorphism** of $X$ if there exists a measurable partition $\mathcal{P} = \{A_1, \dots, A_n\}$ of $X$ such that $\mu(A_i) > 0$, $T|_{A_i} \equiv T_i$ is one-to-one, $T_1$ is one-to-one and onto $X$, and each $A_i$ is maximal with respect to $\mu$ in $X \setminus \bigcup_{j<i} A_j$.

*Definition 2.2:* $T$ is **conservative** (with respect to $\mu$) if for every $A \in \mathcal{B}$ of positive measure, there exists an $m \in \mathbf{N}$ such that $\mu(T^{-m}(A) \cap A) > 0$. $T$ is **ergodic** (with respect to $\mu$) if for every $A \in \mathcal{B}$ such that $T^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

*Definition 2.3* (The Jacobian and the Radon–Nikodým derivative): Given a nonsingular bounded-to-one endomorphism $(X, \mathcal{B}, \mu; T)$, for each $x \in A_i$ let

$$J_\mu T(x) = \frac{d\mu T_i}{d\mu}(x).$$

We set $J_\mu T(x) = 0$ for all $x \in X \setminus \bigcup_i A_i$. This is called the **Jacobian** of $T$ with respect to $\mu$.

The (global) **Radon–Nikodým derivative** of $T$, denoted $\omega_\mu$, is defined to be the unique $T^{-1}\mathcal{B}$-measurable function satisfying

$$\int_X f \circ T \cdot \omega_\mu d\mu = \int_X f d\mu \quad \text{for all } f \in L^1(X, \mu).$$

The higher derivatives are defined for each $n \in \mathbf{N}$ by $\omega_\mu(n, x) = \prod_{i=0}^{n-1} \omega_\mu(T^i x)$. The measure $\mu$ is said to be **recurrent for $T$** if

$$\sum_n \omega_\mu(n, x) = \infty \quad \mu\text{-a.e.}$$

We have the following easily verified identity linking the two types of derivatives:

$$\omega_\mu(x) = \left( \sum_{y \in T^{-1}Tx} \frac{1}{J_\mu T(y)} \right)^{-1}. \tag{1}$$

When $T$ is invertible, all measures are recurrent and the Jacobian and the Radon–Nikodým derivative are the same function. It is an open problem as to whether every endomorphism admits an equivalent recurrent measure. In general it is difficult to determine whether a given ergodic measure is recurrent; an invariant measure $\mu$ is always recurrent since $\omega_\mu(x) \equiv 1$ in this case.

The following theorem represents a compilation of results.

PROPOSITION 2.1 (cf. [28]): *Suppose $T$ is countable-to-one and preserves a $\sigma$-finite measure $\nu$ equivalent to $\mu$, then the following are equivalent:*

1. *$\mu$ is recurrent;*
2. *$\phi = d\nu/d\mu$ is a $T^{-1}\mathcal{B}$-measurable density function;*
3. *$\omega_\mu$ is a coboundary; i.e., $\omega_\mu = \frac{\phi}{\phi \circ T}$ $\mu$-a.e.*

*Proof:* (3) $\Rightarrow$ (2): Since by definition $\omega_\mu$ is $T^{-1}\mathcal{B}$-measurable, so is the quotient $\frac{\phi}{\phi\circ T}$. Obviously $\phi \circ T$ is $T^{-1}\mathcal{B}$-measurable, so the product of $\omega_\mu$ with $\phi \circ T$, which is $\phi$, is $T^{-1}\mathcal{B}$-measurable as well. (2) $\Rightarrow$ (3) follows since $\omega_\nu = 1$ and

$$\frac{\omega_\nu}{\omega_\mu} = \frac{\phi \circ T}{E(\phi | T^{-1}\mathcal{B})},$$

where $E(\cdot|\mathcal{A})$ denotes the usual conditional expectation onto $\mathcal{A} \subseteq \mathcal{B}$. (3) $\Rightarrow$ (1) since coboundaries are easily shown to be recurrent (see [11]). (1) $\Rightarrow$ (3) is proved in [28]. ∎

*Definition 2.4:* Let $T$ be an endomorphism of a Lebesgue space $(X, \mathcal{B}, \mu)$ such that $\mu$ is ergodic, conservative, and nonsingular. We call $\mu$ a **type $II_1$ measure for $T$** if it is absolutely continuous with respect to some invariant probability measure. When $X$ is a Riemannian manifold, and $\mu$ is an invariant probability measure which is absolutely continuous with respect to the volume form, then we call $\mu$ an **acip**. We call $\mu$ a **type *III* measure for $T$** or we say $T$ is a **type** *III* endomorphism if $T$ admits no $\sigma$-finite invariant measure equivalent to $\mu$.

In this paper many examples deal with unimodal maps. A map $f: I \to I$, $I = [0,1]$, is called **unimodal**, if there exists a unique point, $c$, the **critical point**, such that $f|_{[0,c)}$ is increasing and $f|_{(c,1]}$ is decreasing. We write $c_n := f^n(c)$. The forward orbit of a point $x$ is denoted as $\mathrm{orb}(x)$. A unimodal map is onto on the

**dynamical core**, i.e. the interval $[c_2, c_1]$. Therefore we will always restrict to this interval. A unimodal map $f$ is bounded-to-one; Definition 2.1 is satisfied by taking $A_1 = (c, c_1]$ and $A_2 = [c_2, c)$. For $x \neq c$, let the **symmetric point** $\hat{x}$ be the point such that $\hat{x} \neq x$ and $f(x) = f(\hat{x})$. Restricted to the dynamical core, $\hat{x}$ is only defined if $x \in [c_2, \hat{c}_2] \setminus \{c\}$. We call a Radon–Nikodým $g$ derivative **symmetric** if $g(x) = g(\hat{x})$ for all $x \in [c_2, \hat{c}_2] \setminus \{c\}$.

It can happen that there exists an interval $J \ni c$, $J \neq [c_2, c_1]$, such that $f^n(J) \subset J$ for some $n \geq 1$. In this case $f$ is called **renormalizable**. Take $J$ maximal and $n$ minimal with these properties. Then $J$ is called a **restrictive** or **periodic** interval of period $n$. An appropriate affine rescaling of $f^n|_J$, called the **renormalization**, is again a unimodal map, which can be renormalizable or not. Therefore we can distinguish between infinitely renormalizable and finitely renormalizable maps. In the latter case, the deepest, i.e. the last renormalization, is itself nonrenormalizable.

## 3. Non-recurrent measures

In this section we consider the family of tent maps on $I$, defined by

$$T_a(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{2}, \\ a(1-x) & \text{if } x \geq \frac{1}{2}, \end{cases}$$

and we take the slope $a \in (1, 2]$. $T_2$ is the **full tent map**. The critical point is $c = \frac{1}{2}$ and the interval $[T_a^2(c), T_a(c)] = [(2-a)a/2, a/2]$ is the dynamical core. It is easily verified that $T_a$ is renormalizable (of period 2) if and only if $a \leq \sqrt{2}$. The renormalization is the tent map with slope $a^2$. Therefore $T_a$ is at most finitely renormalizable for $a > 1$. We put the normal Borel structure on the dynamical core, and let $m_a$ be the normalized Lebesgue measure on it. In this setting $T_a$ is bounded-to-one, and the partition $\zeta = \{(c, c_1], [c_2, c)\}$ generates the $\sigma$-algebra of Borel sets under $T_a$. $T_a$ is clearly nonsingular. It is well known that $T_a$ is ergodic with respect to $m_a$, and also conservative, provided it is nonrenormalizable.

We can compute $\omega_{m_a}(x)$ explicitly from equation (1). At points where the map $T_a$ is one-to-one, (i.e. $x = T_a^{-1} T_a x$) we have $\omega_{m_a}(x) = a$; at the points where $T_a$ is two-to-one, (i.e. $\{x, \hat{x}\} = T_a^{-1} T_a x$, $x \neq \hat{x}$) we have $\omega_{m_a}(x) = a/2$. We do not define $\omega_{m_a}(c)$. We compute the higher derivatives to be

$$\omega_{m_a}(n, x) = a^n 2^{-r(n,x)},$$

where

$$r(n, x) = \#\{0 \leq i < n; T_a \text{ is two-to-one at } T^i(x)\}.$$

Since $T_a$ admits an ergodic acip $\nu_a \sim m_a$, it follows that

$$\lim_{n\to\infty} \frac{r(n,x)}{n} \text{ exists and is constant } \nu_a\text{-a.e.}$$

Define the sequence $\theta_i \in \{-1,1\}^{\mathbf{N}}$ as follows:

$$\theta_n = \begin{cases} 1 & \text{if } \#\{2 \le i \le n; c_i > 1\} \text{ is even,} \\ -1 & \text{if } \#\{2 \le i \le n; c_i > 1\} \text{ is odd.} \end{cases}$$

We have the following theorem describing the density of $\nu_a$:

THEOREM 3.1 ([3]): *The Radon–Nikodým derivative*

$$\frac{d\nu_a}{dm_a}(x) = \varphi(x) = \sum_{\substack{n\ge 1 \\ c_{n+1}<x<c_1}} \frac{\theta_n}{a^n}.$$

We use this result to prove the following:

THEOREM 3.2: *There are only two values of $a$ such that $T_a$ has a symmetric density function for $\nu_a$. Hence Lebesgue measure is not recurrent for tent maps, except for the slopes $a = 2$ and $a = \sqrt{2}$.*

*Proof:* We divide the proof into two steps. First we claim that the set orb$(c)$ is not $T_a^{-1}\mathcal{B}$-measurable, unless $a = 2$ or $a = \sqrt{2}$. Indeed, suppose it were, then $\hat{c}_m \in \text{orb}(c)$ for every $m \in \mathbf{N}$. So there exists $n$ such that $c_n = \hat{c}_m$. In particular, if $a \neq 2$, there exists $n > 2$ such that $c_n = \hat{c}_2$. Then $c_{n+1} = c_3$, whence $c_3$ is $n-2$-periodic. Setting $n = 3$, we see that this can occur with $c_3$ a fixed point. In this case orb$(c) = \{c_1, c_2, c_3 = \hat{c}_2\}$ is indeed symmetric. This is met when $a = \sqrt{2}$. Using Theorem 3.1, one can show that $d\nu_a/dm_a$ is constant on $(c_2, \hat{c}_2)$ for both $a = \sqrt{2}$ and $a = 2$. Hence by Proposition 2.1, $\mu_a$ is recurrent. We show that nothing else can occur.

If $n > 3$, then $c_n \neq \hat{c}_2$. If $c_3 \notin [c_2, \hat{c}_2]$, then $c_3 = (2-a)a^2/2 > \frac{1}{2}(2-2a+a^2) = \hat{c}_2$. This in turn implies that $a^3 - a^2 - 2a + 2 < 0$, which is impossible for $a > \sqrt{2}$. If $a < \sqrt{2}$, then $T_a$ is renormalizable: $T_a^2([c_2, \hat{c}_2]) \subset [c_2, \hat{c}_2]$ and $[c_2, \hat{c}_2] \cap [c_3, c_1] = \emptyset$. Now $c_4 \in (c_2, \hat{c}_2)$, and $\hat{c}_4 = c_{n'}$ for some $n' > 4$. But this is impossible because $c_3$ is periodic. The remaining possibility is $c_3 \in (c_2, \hat{c}_2)$, but then also $\hat{c}_3 = c_{n'}$ for some $n' > 3$. This again is impossible because $c_3$ is periodic.

In the second step of the proof we show that $\varphi$ is not a symmetric density if orb$(c)$ is not symmetric. Let $c_m$ be such that $\hat{c}_m \notin \text{orb}(c)$, but exists. The function

$$\varphi(x) = \sum_{\substack{n\ge 1 \\ c_{n+1}<x<c_1}} \frac{\theta_n}{a^n}$$

clearly has a discontinuity at $c_m$, with a jump of size $1/a^{m-1}$. Choose $k$ satisfying

$$\frac{1}{a^k}\frac{a}{a-1} < \frac{1}{a^m} =: \varepsilon,$$

and choose neighbourhoods $U$ of $c_m$ and $\hat{U}$ of $\hat{c}_m$ such that $c_i \notin U \cup \hat{U}$ for all $i \leq k$. Then it follows that $\sup_{x,y\in U} |\varphi(x) - \varphi(y)| > \varepsilon$, while $\sup_{x,y\in\hat{U}} |\varphi(x) - \varphi(y)| \leq \varepsilon$. Hence $\varphi$ cannot be symmetric. This concludes the proof of the theorem. ∎

*Remark:* A similar situation may hold for differentiable families, and in particular the quadratic family $f_b(x) = bx(1-x)$. For the full quadratic map, i.e. $b = 4$, the invariant density is known to be

$$\frac{d\mu_b}{dm}(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}},$$

which is clearly symmetric. For $b = 3.67857\ldots$, the parameter value corresponding to the tent map with slope $\sqrt{2}$, the density is not symmetric, see [8]. Therefore Lebesgue measure is nonrecurrent for this value.

For all other values of $b \in [0,4]$ such that $f_b$ has an acip, we expect Lebesgue measure to be nonrecurrent. We outline why this should be true. Using the same proof as for the tent maps, we can show that the critical orbit is not symmetric. If an acip exists, its density function will have a pole at every forward image of $c$, cf. [18]. For example, if $f_b$ additionally satisfies the Collet–Eckmann condition (i.e. $\liminf_n \frac{1}{n}\log Df^n(c_1) > 0$), the density has the form

$$\frac{d\mu_b}{dm}(x) = g(x)\sum_{i\geq 1}\lambda^r|x-c_r|^{\frac{1}{2}}$$

for some $\lambda \in (0,1)$ and a function $g$ of bounded variation (see [19]). Whenever the closure of the critical orbit is nonsymmetric, this density is clearly nonsymmetric.

## 4. Hamachi measure

The starting point for all the constructions in this paper is an example of Hamachi from 1981 [9]. He constructed a measure $\mu$ for the two-sided shift space $X = \{0,1\}^{\mathbf{Z}}$ with the usual Borel structure with the following properties.

THEOREM 4.1 ([9]): *There exists a Borel measure $\alpha$ for the shift $\sigma$ on $X = \{0,1\}^{\mathbf{Z}}$ such that:*

1. *$\alpha$ is a product measure on $X$;*
2. *$\alpha$ is nonsingular, conservative, and ergodic for $\sigma$;*
3. *$\alpha$ is a type III measure for $\sigma$.*

Later it was shown by Dajani and Hawkins that a one-sided version of Hamachi measure gives a type *III* one-sided shift [4]. This observation about the Hamachi measure was also made independently by Silva and Thieullen [28]. We state the precise result since it is the basic measure on which further constructions are based.

COROLLARY 4.1 ([4]): *There exists a Borel measure for the shift on $X^+ = \{0,1\}^{\mathbf{N}}$ such that:*

1. *$\alpha$ is a product measure on $X^+$;*
2. *$\alpha$ is nonsingular, conservative, and ergodic for $\sigma$;*
3. *$\alpha$ is recurrent for $\sigma$;*
4. *$\alpha$ is a type III measure for $\sigma$.*

We give a brief description of the type *III* measure of Hamachi here.

4.1 A DESCRIPTION OF THE HAMACHI PRODUCT MEASURE. We define the measure $\alpha$ to be of the form $\alpha = \prod_{n\geq 1}\alpha_n$ by specifying each factor measure $\alpha_n$. We begin by defining some measures on the 2-point set $\{0,1\}$:

1. We denote by $\beta$ the equally distributed measure $\beta(0) = \beta(1) = \frac{1}{2}$. If we put the measure $\beta$ on each factor of $X^+$, we will denote this Bernoulli measure by $\hat{\beta}$.

2. For a sequence $\lambda_k$ such that $\lambda_1 > \lambda_2 \cdots > 1$ (to be chosen by induction later), we define a measure $\gamma_k(0) = 1/(1+\lambda_k)$ and $\gamma_k(1) = \lambda_k/(1+\lambda_k)$.

We will define sequences of integers $\{M_k\}_{k\geq 1}$, $\{N_k\}_{k\geq 1}$, satisfying: $M_1 > 1$ is arbitrary, also $N_1 > M_1$ is arbitrary, and $N_k = M_k + n_k$, $M_{k+1} = N_k + m_k$. Here, $n_k$ and $m_k$ are positive integers chosen inductively, with the inductive step outlined below.

We then define

$$\alpha_n = \begin{cases} \beta & \text{if } 0 < n \leq M_1, \\ \gamma_k & \text{if } M_k < n \leq N_k, \\ \beta & \text{if } N_k < n \leq M_{k+1}. \end{cases}$$

As was shown in [15], the measure $\mu$ is nonsingular for the shift if and only if

$$\sum_{k=1}^{\infty}(\log\lambda_k)^2 < \infty. \tag{2}$$

For later purposes we will choose the $\lambda_k$'s such that $\prod_{k=1}^{\infty}\lambda_k < 1.1$. Clearly formula (2) is still satisfied. We add another condition on the choice of $\lambda_k$ in the inductive step.

Hamachi gives an inductive algorithm for choosing the sequence $(\lambda_k, n_k, m_k)_{k\geq 1}$ so that the shift is nonsingular, ergodic, and type *III* with respect to $\alpha$. Since we will modify the measure later, we will outline the inductive step, omitting the details since they are carefully written out in [9].

*A. Starting the inductive argument:* We choose $\lambda_1 > 1$ to be arbitrary; also $n_1 \in \mathbf{N}$ is arbitrary (greater than 1) and $m_1 \in \mathbf{N}$ is any number such that $m_1 > n_1 + 1$. Choose any decreasing sequences of positive numbers $\{p_k\}$ and $\{\varepsilon_k\}$, $k \geq 1$, such that

1. As $k \to \infty$, $p_k \to 0$ and $\varepsilon \to 0$;
2. $\sum_{k=1}^{\infty} p_k = \infty$;
3. $\sum_{k=1}^{\infty} \varepsilon_k < \infty$;
4. Define $\eta_k = \sum_{t=k}^{\infty} \varepsilon_t$ (the tail of the $\varepsilon_k$ series).

*B. The inductive choice of $\lambda_k$ in order to introduce the distortion from the $(\frac{1}{2}, \frac{1}{2})$ measure $\hat{\beta}$:* Keeping in mind that if $\lambda_k = 1$ for all $k$, then we have the $\beta$ measure on each factor, and preserving the nonsingularity condition given by formula (2), we choose $1 < \lambda_k < \lambda_{k-1}$ so that

$$(2\lambda_k/1+\lambda_k)^{M_{k-1}} < \lambda_k^{M_{k-1}} < e^{\varepsilon_k}.$$

We also choose $\rho_k > 0$ such that

$$1 < (\lambda_1)^{2M_{k-1}} < (\lambda_k)^{\rho_k}.$$

*C. The inductive choice of $n_k$, the integer which determines how long we must distort the measure by $\lambda_k$:* We consider for the moment all possible cylinders of length $N_k = M_k + n_k$; and we note that the measure of all such cylinders with exactly $t$ 1's occurring somewhere between $M_k$ and $N_k$ is given by the binomial distribution formula

$$f_k(t) = \binom{n_k}{t}\left(\frac{\lambda_k}{1+\lambda_k}\right)^t\left(\frac{1}{1+\lambda_k}\right)^{n_k-t},$$

$t = 0, 1, \ldots, n_k$. We choose $n_k$ large enough so that $f_k(t)$ is the "correct size for enough of the $t$'s enough of the time" in order to reflect the fact that we have changed from the $(\frac{1}{2}, \frac{1}{2})$ measure to the $(\frac{1}{1+\lambda_k}, \frac{\lambda_k}{1+\lambda_k})$ measure. In particular, if we solve for $c_k > 0$ so that

$$\frac{1}{\sqrt{2\pi}}\int_{-c_k}^{c_k} e^{(-s^2/2)}ds = p_k,$$

then we apply the Central Limit Theorem and choose $n_k$ large enough so that

$$\sum_{|t-\frac{n_k\lambda_k}{1+\lambda_k}|<\frac{\sqrt{n_k\lambda_k}c_k}{1+\lambda_k}-2\rho_k} f_k(t) > \frac{p_k}{4}.$$

*D. The inductive choice of $m_k$, the integer which determines how long we must spend back at the $\beta$ measure for conservativity:* There are two conditions that determine our choice at this step. We choose $m_k$ large enough so that

$$\frac{N_k e^{2\eta_{k+1}}\lambda_1^{3N_k}}{m_k - N_k} < \frac{\varepsilon_k}{2}.$$

This maintains conservativity of $\alpha$; but in addition, in order to ensure that the product measure $\alpha$ is of type *III* (and not just equivalent to an infinite invariant measure), we need to choose $m_k$ large enough so that we can obtain some correct Birkhoff Ergodic Theorem averages with respect to the measure $\hat{\beta}$ on most of the space $X^+$ (on a set of $\hat{\beta}$ measure $> 1-\varepsilon_k$) for certain real-valued functions. We refer the reader to [9] for the details of this inductive step.

For the invertible shift, the measure is extended to the negative indices by setting $\alpha_n = \beta$ if $n \leq 0$. We will call the original two-sided measure constructed by Hamachi $\hat{\alpha}$, and it is easy to show that the noninvertible measure $\alpha$ constructed above satisfies

$$\frac{d\hat{\alpha}\sigma^i}{d\hat{\alpha}}(\ldots,x_{-1},x_0,x) = \omega_\alpha(i,x) \quad \text{for all } x \in X^+, \quad \text{and} \quad x_j \in \{0,1\}, \quad j \leq 0.$$

The choice of $m_k$ will insure that

$$\sum_{i=0}^{\infty} \frac{d\hat{\alpha}\sigma^i}{d\hat{\alpha}}(x) = \infty$$

for the invertible shift, giving conservativity, and for the noninvertible one we have

$$\sum_{i=0}^{\infty} \omega_\alpha(i,x) = \infty.$$

Therefore $\alpha$ constructed in this way is a recurrent measure.

*Definition 4.1:* We will call the measure $\alpha$ constructed in this way, using [9], the **Hamachi** (*type III*) **measure**.

### 4.2 Some variations on the Hamachi measure.

Given

$$X_d^+ = \prod_{n\in\mathbf{N}} \{0,1,\ldots,2d-1\}_n,$$

we define a product measure $\alpha_d$ closely related to the Hamachi measure on $X^+$ as follows.

Choose any positive numbers $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$ such that

$$\sum_{k=1}^{d} p_k = \sum_{k=1}^{d} q_k = \frac{1}{2}.$$

Since the Borel structure on $X_d^+$ is generated by cylinders, we define $\alpha_d$ by specifying its values on cylinder sets. We note first that the $p_i$'s and $q_i$'s determine a measure on $X_d^+$ by

$$P = \prod_{n \in \mathbf{N}} P_n,$$

with

$$P_n(0) = p_1, \quad P_n(1) = p_2, \quad \ldots, \quad P_n(d-1) = p_d,$$
$$P_n(d) = q_1, \quad \ldots, \quad P_n(2d-1) = q_d$$

for each $n \in \mathbf{N}$. The measure $P$ is a Bernoulli measure preserved by the shift $\sigma$, so $\omega_P(x) = 1$ for all $x \in X_d^+$. Each cylinder $C_{e_1 \ldots e_n}$ of length $n$ lies in a dyadic cylinder of length $n$, by recoding each $e_k$ into a 0 if $0 \le e_k < d$ or as a 1 if $d \le e_k < 2d$. We denote the coding map from $2d$ symbols to 2 symbols by $\pi$; $\pi$ is defined pointwise in the obvious way. We now define

$$\alpha_d(C_{e_1 \ldots e_n}) = P(C_{e_1 \ldots e_n}) \frac{\alpha(\pi C_{e_1 \ldots e_n})}{\hat{\beta}(\pi C_{e_1 \ldots e_n})},$$

where $\alpha$ is the Hamachi measure constructed above and $\hat{\beta}$ is the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure. With this definition, we have linearly rescaled the Hamachi measure to any even number of states, so that $\alpha_d = P \cdot \frac{\alpha}{\hat{\beta}} \circ \pi$.

Letting $\sigma_k$ denote the shift on the $k$-symbol space, we have the following lemma.

LEMMA 4.1: *The Radon Nikodým derivative $\omega_{\alpha_d}(x)$ for $\sigma_{2d}$ equals $\omega_\alpha(\pi x)$ for $\sigma_2$.*

*Proof:* We note that since $J_{\alpha_d}\sigma_{2d}(x) = \frac{1}{2} J_P \sigma_{2d}(x) J_\alpha \sigma_2(\pi x)$, then it is easy to compute that

$$\omega_{\alpha_d} = \left( \sum_{y \in \sigma_{2d}^{-1} \sigma_{2d} x} \frac{1}{J_{\alpha_d} \sigma_{2d}(y)} \right)^{-1} = \omega_\alpha(\pi x). \qquad \blacksquare$$

*Remark:* Choose any two sequences of positive numbers $\{p_k\}$ and $\{q_k\}$ such that
$$\sum_{k=1}^{\infty} p_k = \sum_{j=1}^{\infty} q_j = \frac{1}{2}.$$
The $p_k$'s and $q_j$'s determine a probability measure on
$$X_{\mathbf{N}}^{+} = \prod_{n=0}^{\infty} \{0, 1, \ldots, 2k, 2k+1, \ldots\}_n$$
by $P = \prod_{n\geq 1} P_n$, with $P_n(2k) = p_k$, $P_n(2k+1) = q_k$, for each $k \geq 0$ for each $n$. We define a factor map $\pi$ from $X_{\mathbf{N}}^{+}$ onto the dyadic $X^+$ above by
$$\pi(x) = (x_1(\bmod 2), x_2(\bmod 2), \ldots, ).$$
We can consider the usual one-sided shift map $\sigma_{\mathbf{N}}$ on $X_{\mathbf{N}}^{+}$. If we define the measure
$$\alpha_{\mathbf{N}}(C_{e_1\ldots e_n}) = P(C_{e_1\ldots e_n}) \frac{\alpha(\pi C_{e_1\ldots e_n})}{\hat{\beta}(\pi C_{e_1\ldots e_n})},$$
where $\alpha$ is Hamachi measure and $C_{e_1\ldots e_n}$ is any cylinder of length $n$, then $\omega_{\alpha_{\mathbf{N}}}(x) = \omega_\alpha(\pi x)$.

## 5. Hamachi measure for circle maps

In this section we construct the basic differentiable example, on which the other examples are built. A map $f$ on a metric space $X$ (endowed with metric $\rho$) is called **expanding** if there exists $C > 1$ such that $\rho(f(x), f(y)) \geq C\rho(x, y)$ for all $x, y \in X$.

THEOREM 5.1: *There exists a $C^1$ expanding circle map which is type III with respect to Lebesgue measure $m$. Furthermore, $m$ is recurrent for this map.*

*Proof:* Let $S$: $\mathbf{S}^1 \to \mathbf{S}^1$, $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$, be the ordinary angle doubling map: $x \mapsto 2x(\bmod 1)$. $S$ preserves Lebesgue measure $m$. Let $\alpha$ be Hamachi measure on $\Sigma = \{0,1\}^{\mathbf{N}}$. Because $(\Sigma, \sigma)$ is measure-theoretically isomorphic to $(\mathbf{S}^1, S)$, $S$ is type *III* with respect to the measure $\mu$ which is induced by $\alpha$. We fix an orientation on $\mathbf{S}^1$, and define $h$: $\mathbf{S}^1 \to \mathbf{S}^1$ as $h(x) = \mu([0, x))$. As $\mu$ is nonatomic and its support is the whole circle, $h$ is indeed a homeomorphism, and the measure $\mu \circ h^{-1}$ is Lebesgue measure. Therefore $f := h \circ S \circ h^{-1}$ is a type *III* circle map with respect to Lebesgue measure. We will analyze $h$ in detail to show that $f$ can be a Lipschitz map. Using a slight perturbation of $h$, we can obtain an expanding $C^1$ map which is type *III* with respect to Lebesgue measure.

Let $\{M_k\}_{k\geq 1}$, $\{N_k\}_{k\geq 1}$, $1 < M_k < N_k < M_{k+1} < N_{k+1}\cdots$, and $\{\lambda_k\}_{k\geq 1}$, $\lambda_1 > \lambda_2 > \cdots > 1$, be the sequences appearing in the construction of the Hamachi product measure. Recall that $\prod_k \lambda_k < \frac{11}{10}$. Let $\varepsilon = \{\varepsilon_k\}$ be a sequence of nonnegative reals. If $\varepsilon_k \equiv 0$, then the construction below yields $h(x) = \mu([0,x))$. Otherwise we will choose $\varepsilon_k > 0$ inductively (with $\varepsilon_1 = 0.1$ and $\varepsilon_k \searrow 0$) to obtain a $C^1$ degree 2 circle map $f_\varepsilon = h_\varepsilon \circ S \circ h_\varepsilon^{-1}$. For each $k \in \mathbf{N}$ and $\eta \in (-\frac{1}{2}, \frac{1}{2})$ let $\psi_{k,\eta}$ (satisfying $\psi_{k,\eta}(0) = 0$) be the map with a piecewise linear derivative:

$$D\psi_{k,\eta}(x) = \begin{cases} 1 + \frac{x}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in [0, \varepsilon_k), \\ \frac{2}{1+\lambda_k} & \text{if } x \in [\varepsilon_k, \frac{1}{2} + \eta - \varepsilon_k], \\ 1 - \frac{x - \frac{1}{2} - \eta}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in (\frac{1}{2} + \eta - \varepsilon_k, \frac{1}{2} + \eta + \varepsilon_k), \\ \frac{2\lambda_k}{1+\lambda_k} & \text{if } x \in [\frac{1}{2} + \eta + \varepsilon_k, 1 - \varepsilon_k], \\ 1 + \frac{x-1}{\varepsilon_k}\frac{1-\lambda_k}{1+\lambda_k} & \text{if } x \in (1 - \varepsilon_k, 1]. \end{cases}$$

Then

$$\int_0^1 D\psi_{k,\eta}(x)dx = 1 + 2\eta\frac{1-\lambda_k}{1+\lambda_k},$$

and $\psi_{k,\eta}(I) = I$ if $\eta = 0$. Moreover, $D\psi_{k,\eta}(0) = D\psi_{k,\eta}(1) = 1$. This is necessary to glue these maps together and still have a $C^1$ diffeomorphism.

Let $C_{e_1e_2\cdots e_n} \subset \mathbf{S}^1$ be an $n$-cylinder, labelled by the first $n$ coordinates of its itinerary: $C_0 = [0, \frac{1}{2})$, $C_1 = [\frac{1}{2}, 1)$, $C_{01} = [\frac{1}{4}, \frac{1}{2})$, and so on. Therefore $\mathbf{S}^1 = \bigcup_{e_1e_2\cdots e_n \in \{0,1\}^n} C_{e_1e_2\cdots e_n}$. Define $h_n$ as follows:

- If $n \leq M_1$ or $N_k < n \leq M_{k+1}$ for some $k$, then $h_n$ is the identity.
- If $M_k < n \leq N_k$ for some $k$, then $h_n$ is made up of $2^{n-1}$ scalings of $\psi_{k,\eta}$. Let $\xi = h_{n-1} \circ \cdots \circ h_1(\zeta)$, where $\zeta$ is the midpoint of $C_{e_1\cdots e_{n-1}}$. Let

$$[x,y) = h_{n-1} \circ \cdots \circ h_1(C_{e_1\cdots e_{n-1}}) \quad \text{and} \quad \eta_n = \frac{\xi - x}{y - x} - \frac{1}{2}.$$

(Note that if $\varepsilon_k \equiv 0$, $h_{n-1} \circ \cdots \circ h_1$ is linear on $C_{e_1\cdots e_{n-1}}$ and $\eta_n \equiv 0$.) Let $\tilde{h}_n(0) = 0$, and assuming inductively that $\tilde{h}_n$ is defined on the cylinder $[x', y')$ to the left of $[x,y)$, we define for $z \in [x,y)$

$$\tilde{h}_n(z) = \lim_{z' \nearrow y'} \tilde{h}_n(z') + (y-x)\psi_{k,\eta_n}\Big(\frac{z-x}{y-x}\Big). \tag{3}$$

Finally let $h_n(z) = t_n\tilde{h}_n(z)$, where $t_n > 0$ is such that $t_n \int_0^1 D\tilde{h}_n(z)dz = 1$.

Let $H_n = h_n \circ \cdots \circ h_1$ and $h_\varepsilon = \lim_n H_n$. We will see below that $t_n \to 1$ exponentially. This will imply that $\lim_n H_n$ exists pointwise. Define also $f_n =$

$H_n \circ S \circ H_n^{-1}$. We need to show that for $\{\varepsilon_k\}$ well chosen, $\{f_n\}$ is a convergent sequence in the $C^1$ topology.

Suppose that we have chosen $\varepsilon_l$ for $l < k$ and take $M_k < n \leq N_k$. Let us first derive some estimates for $Df_{N_k}$ which hold if $\varepsilon_k = 0$. Then we choose the value of $\varepsilon_k$ for the inductive step in formula (4) using a continuity argument.

By the **relative displacement** of a point $c \in (a,b)$ **due to** a homeomorphism $h$ we mean

$$\frac{h(c)-h(a)}{h(b)-h(a)} - \frac{c-a}{b-a}.$$

Each number $\eta_n$ measures the relative displacement of the midpoint of (one of the) $C_{e_1\cdots e_{n-1}}$ due to $H_{n-1}$. We estimate $\eta_n$ under the temporary assumption $\varepsilon_k = 0$. We only have to consider $h_m$ for $m \leq N_{k-1}$, since for all $N_{k-1} < m < n$, the homeomorphisms $h_m$ are piecewise affine on $H_{m-1}(C_{e_1\cdots e_{n-1}})$ for $N_{k-1} < m < n$. For $m \leq N_{k-1}$, the interval $H_{m-1}(C_{e_1\cdots e_{n-1}})$ is exponentially small compared to $H_{m-1}(C_{e_1\cdots e_{m-1}})$. Indeed, because the distortions

$$\frac{\sup Dh_l}{\inf Dh_l} \leq \frac{6}{5} \quad \text{for all } l \in \mathbf{N}$$

and

$$\frac{M_k}{N_{k-1}} \gg 1,$$

$|H_{m-1}(C_{e_1\cdots e_{n-1}})| \leq 1.9^{-(n-m)}|H_{m-1}(C_{e_1\cdots e_{m-1}})|$. Hence the relative displacement of any $c \in H_{m-1}(C_{e_1\cdots e_{n-1}})$ due to $h_m$ is $\leq \mathcal{O}((1.9)^{-(n-m)})$. This gives a relative displacement due to $H_{n-1}$ (using $M_k/N_{k-1} \gg 1$)

$$\eta_n \leq \sum_{m=1}^{N_{k-1}} \mathcal{O}((1.9)^{-(n-m)}) \leq (1.8)^{-n}.$$

By the same argument $\eta_n \geq -(1.8)^{-n}$. Because this is true for every cylinder $C_{e_1\cdots e_{n-1}}$ and

$$\int_0^1 D\psi_{k,\eta_n}(z)dz = 1 + 2\eta_n \frac{1-\lambda_k}{1+\lambda_k},$$

it follows also that $1 - \mathcal{O}((1.8)^{-n}) \leq t_n \leq 1 + \mathcal{O}((1.8)^{-n})$.

Clearly

$$Df_n(x) = \frac{Dh_n(f_{n-1}(y))}{Dh_n(y)} Df_{n-1}(y),$$

where $y = h_n^{-1}(x)$. Now setting $n = N_k$ and $y_i = h_i^{-1} \circ \cdots \circ h_{N_k}^{-1}(x)$, we get (because $h_m$ is the identity for $N_{k-1} < m \leq M_k$)

$$\begin{aligned} Df_{N_k}(x) &= Df_{N_{k-1}}(y_{N_{k-1}+1}) \frac{Dh_{N_{k-1}+1}(f_{N_{k-1}}(y_{N_{k-1}+1}))}{Dh_{N_{k-1}+1}(y_{N_{k-1}+1})} \cdots \frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{N_k}(y_{N_k})} \\ &= Df_{N_{k-1}}(y_{N_{k-1}+1}) \frac{Dh_{M_k+1}(f_{M_k}(y_{M_k+1}))}{Dh_{M_k+1}(y_{M_k+1})} \cdots \frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{N_k}(y_{N_k})}. \end{aligned}$$

By construction, $f_n(h_n^{-1}(z))$ lies in the left part of $H_{n-2}(C_{e_2 \cdots e_{n-1}})$ if and only if $z$ lies in the left part of $H_{n-1}(C_{e_1 \cdots e_{n-1}})$, for all $M_k < n \leq N_k$. Therefore

$$\frac{|Dh_{n-1}(f_{n-2}(y_{n-1}))|}{|Dh_n(y_n)|} = \frac{t_{n-1}}{t_n},$$

and hence

$$\begin{aligned} Df_{N_k}(x) &= Df_{N_{k-1}}(y_{N_{k-1}+1}) \frac{Dh_{N_k}(f_{N_k-1}(y_{N_k}))}{Dh_{M_k+1}(y_{M_k+1})} \prod_{n=M_k+2}^{N_k} \frac{t_{n-1}}{t_n} \\ &\leq Df_{N_{k-1}}(x) \frac{Df_{N_{k-1}}(y_{N_{k-1}+1})}{Df_{N_{k-1}}(x)} \frac{D\tilde{h}_{N_k}(f_{N_k-1}(y_{N_k}))}{D\tilde{h}_{M_k+1}(y_{M_k+1})} \\ &\leq Df_{N_{k-1}}(x)(1 + \mathcal{O}((1.8)^{-M_k}))\lambda_k, \end{aligned}$$

because $y_{N_{k-1}+1}$ and $x$ lie in the same cylinder $H_{N_{k-1}}(C_{e_1 \cdots e_{M_k-1}})$ which (as we saw before) is exponentially small compared to $H_{N_{k-1}}(C_{e_1 \cdots e_{N_{k-1}-1}})$. A similar argument gives

$$Df_{N_k}(x) \geq Df_{N_{k-1}}(x) \frac{1}{\lambda_k}(1 - \mathcal{O}((1.8)^{-M_k})).$$

These derivatives depend continuously on $\varepsilon_k$. We choose $\varepsilon_k > 0$ so small that

$$(4)\quad Df_{N_{k-1}}(x) \frac{1}{\lambda_k}(1 - (1.7)^{-M_k}) \leq Df_{N_k}(x) \leq Df_{N_{k-1}}(x)\lambda_k(1 + (1.7)^{-M_k}),$$

for all $x \in \mathbf{S}^1$. Then both $f_{N_k}$ and $Df_{N_k}$ converge uniformly on $\mathbf{S}^1$. The limit $f_\varepsilon$ satisfies

$$2\prod_k \frac{1 - (1.7)^{-M_k}}{\lambda_k} \leq Df(x) \leq 2\prod_k \lambda_k(1 + (1.7)^{-M_k})$$

and therefore is an expanding $C^1$ circle map.

We now check that $f_\varepsilon$ is indeed type *III* with respect to Lebesgue measure. Define $\varphi_{\varepsilon,n}(x) = Dh_n(H_{n-1}(x))$, and if $\varepsilon_k \equiv 0$, we write $\varphi_n(x)$. If $\varepsilon_k \equiv 0$, then $f_\varepsilon$ is Lipschitz continuous, and $\mu$ corresponds to the original Hamachi product measure. For $\varepsilon_k \not\equiv 0$, $h_\varepsilon$ defines a measure by $\mu_\varepsilon := m \circ h_\varepsilon$. It is enough to show that for a good choice of $\varepsilon$ above, $\mu_\varepsilon \sim \mu$. We show that the Radon–Nikodým derivative is

$$\Phi_\varepsilon(x) := \frac{d\mu_\varepsilon}{d\mu}(x) = \prod_{n=1}^{\infty} \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)}.$$

Choosing $\varepsilon_k$ sufficiently small, we can make sure that $\Phi_\varepsilon$ is bounded and bounded away from 0 $\mu$-a.e. Indeed, for $M_k < n \leq N_k$, let

$$A_n = \left\{ x \mid \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)} < 1 - \frac{1}{n^2} \text{ or } \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)} > 1 + \frac{1}{n^2} \right\}.$$

Obviously

$$\prod_n \frac{\varphi_{\varepsilon,n}(x)}{\varphi_n(x)}$$

is finite and positive if $x$ is not too often contained in $A_n$. But because the numbers $\lambda_k$ are the same for $\varphi_n$ and $\varphi_{\varepsilon,n}$, $\mu(A_n) \to 0$ as $\varepsilon_k \to 0$. Extending the choice at inequality (4), take $\varepsilon_k$ so small that $\mu(A_n) \leq 1/n^2$ for all $M_k < n \leq N_k$. Then the set of points visiting an $A_n$ infinitely often has zero measure, because by the Borel–Cantelli Lemma, $\mu(\bigcap_l \bigcup_{n>l} A_n) = 0$. Hence $\mu$ is equivalent to $\mu_\varepsilon$.

It is a property of Hamachi measure that $\mu$ is a recurrent measure for $S$. We claim that $\mu_\varepsilon$ is also recurrent for $S$, which is equivalent to showing that the measure $m$ is recurrent for $f_\varepsilon$. Since $\mu \sim \mu_\varepsilon$, it suffices to show that $\Phi_\varepsilon$ is constant on symmetric points of $S$ ([12]). Suppose that $x$ and $y$ are such that $S(x) = S(y)$. Then $x = y + \frac{1}{2}$ and by the symmetry in the construction of $h_\varepsilon$, $\Phi_\varepsilon(x) = \Phi_\varepsilon(y)$.

*Remark:* Using a Hamachi measure on the one-sided shift on $d$ symbols, and the techniques developed in this section, we can construct $C^1$ expanding type *III* circle maps of any degree $d \geq 2$.

## 6. Hamachi measure for interval maps

In this section we modify the previous construction for tent maps.

THEOREM 6.1: *There exists a unimodal map, conjugate to the full tent map which is type III with respect to Lebesgue measure $m$. This map can be chosen to be $C^1$, or piecewise $C^1$ and expanding.*

*Proof:* It is well-known that $(\Sigma, \sigma)$ is measure-theoretically isomorphic to $(I, T)$, where $T = T_2$ is the full tent map. Indeed, take $a = a_1 a_2 \cdots \in \Sigma$, and let

$\vartheta(a_1 \cdots a_n)$ be the number of ones in $a_1 \cdots a_n$. Let $\tilde{a}$ be defined as

$$\tilde{a}_i = \begin{cases} a_i & \text{if } \vartheta(a_1 \cdots a_{i-1}) \text{ is even,} \\ 1 - a_i & \text{if } \vartheta(a_1 \cdots a_{i-1}) \text{ is odd.} \end{cases}$$

If $x(\tilde{a})$ is the point in $I \setminus \bigcup_{n \le 0} T^n(\frac{1}{2})$ whose itinerary is $\tilde{a}$, then $a \mapsto x(\tilde{a})$ is the required isomorphism. So we can again pull back Hamachi measure $\alpha$, obtaining a measure with respect to which $T$ is type *III*. We proceed as in the circle case; the only adjustment to be made is to change formula (3) into

$$h_n(z) = \lim_{z' \nearrow y'} h_n(z') + \begin{cases} (y-x)\psi_{k,\eta_n}(\frac{z-x}{y-x}) & \text{if } \vartheta(e_1 \cdots e_{n-1}) \text{ is even,} \\ (y-x)(\psi_{k,\eta_n}(1) - \psi_{k,\eta_n}(\frac{y-z}{y-x})) & \text{if } \vartheta(e_1 \cdots e_{n-1}) \text{ is odd.} \end{cases}$$

This will give us the required piecewise $C^1$ and expanding map $f$.

Note that $f'(0) = -f'(1)$. In order to obtain a $C^1$ (but no longer expanding) example, we can do the following. It can be easily checked that $T$ and the quadratic map $Q(x) = 4x(1-x)$ are smoothly conjugate: If $g(x) = \frac{1}{2}(1 - \cos \pi x)$, then $g \circ T = Q \circ g$. Applying the same conjugacy on $f$, we get a $C^1$ map $\tilde{f} = g \circ f \circ g^{-1}$, which is type *III* with respect to the measures $m \circ g^{-1}$ and $m$. ∎

These results allow the following generalization:

THEOREM 6.2: *For every $a \in (1,2]$ such that the tent map $T_a$ has a nowhere dense critical orbit, there exists a map $f$, topologically conjugate to $T_a$, which is type III with respect to Lebesgue measure.*

*Proof:* Assume $a > \sqrt{2}$, because otherwise $T_a$ is renormalizable, and we can consider the deepest renormalization of $T_a$ instead. Let $V \subset [c_2, c_1]$ be an open interval such that $\text{orb}(c) \cap V = \emptyset$. Due to the expansion properties of $T_a$, there exists $n$ minimal such that $c \in T_a^n(V)$. Take any $p \in T_a^n(V)$, $p \neq c$, such that also $\hat{p} \in T_a^n(V)$ and $\text{orb}(p) \cap (p, \hat{p}) = \emptyset$. Let $U = T_a^{-n}((p,\hat{p})) \cap V$. Then $\text{orb}(c)$ and $\text{orb}(\partial U)$ are disjoint from $U$, and $q = T_a^{-n}(c) \cap U$ is the middle point of $U$. It follows that the first return map $F\colon U \to U$ has countably many branches $F\colon J \to U$, $F|_J = T_a^{s(J)}$, all of which are onto. Also $F(q)$ is not defined and $F(q+\varepsilon) = F(q-\varepsilon)$ for all $\varepsilon < \frac{1}{2}|U|$. By the techniques discussed previously, there exists an ergodic nonsingular nonatomic measure $\mu$ with respect to which $F$ is type *III*. Also $\mu(U') > 0$ for every nondegenerate subinterval $U' \subset U$. We can pull back $\mu$ to a measure $\bar{\mu}$ of the original map $T_a$ as

$$\bar{\mu}(B) = \sum_J \sum_{i=0}^{s(J)-1} \mu(T_a^{-i}(B) \cap J),$$

where the first sum is taken over all branch domains of $F$. Due to a result of Silva and Thieullen [28], $T_a$ is type *III* with respect to $\bar{\mu}$. It is easy to show that $\bar{\mu}$ is again ergodic nonatomic and nonsingular, and that $\bar{\mu}(I') > 0$ for every nondegenerate subinterval $I' \subset [c_2, c_1]$. Define again the homeomorphism $h$: $[c_2, c_1] \to [0,1]$ as $h(x) = \bar{\mu}([c_2, x))$. Then $f = h \circ T_a \circ h^{-1}$ is conjugate to $T_a|_{[c_2,c_1]}$, and $f$ is type *III* with respect to Lebesgue measure. ∎

## 7. Hamachi measure for maps on the sphere

In this section we extend our construction to the Riemann sphere, $\mathbf{C} \cup \infty \equiv \mathbf{C}_\infty$. Our example is motivated by a classical construction given by Lattès to construct a rational map of the sphere whose Julia set is the whole sphere [22].

7.1 ANALYTIC TYPE *III* MAPS OF THE TORUS. We begin by extending our construction above to the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ as follows. Let us first remark that the angle doubling map $Sx = 2x \pmod 1$ gives rise to a measure preserving Bernoulli four-to-one map of $\mathbf{T}^2$ by

$$S \times S(x,y) = Q(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} (x,y) \pmod 1.$$

With respect to two-dimensional Lebesgue measure on $\mathbf{T}^2$, denoted $m_2$, this is a one-sided Bernoulli shift of entropy $\log 4$. We can identify $\mathbf{T}^2$ with $\mathbf{S}^1 \times \mathbf{S}^1$ and put $\mu$, the Hamachi measure we constructed in Section 6, on one copy of $\mathbf{S}^1$. The measure we now have on $\mathbf{S}^1 \times \mathbf{S}^1$ is $m \times \mu$, which we denote by $\mu_2$. We have the following result. We will write $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$.

THEOREM 7.1: *The map $Q = S \times S$ on $(\mathbf{S}^1 \times \mathbf{S}^1, \mathcal{B}_2, \mu_2)$ satisfies:*

1. *$Q$ is a 4-to-1 map with respect to $\mu_2$ ;*
2. *$Q$ is continuous, nonsingular, conservative, and ergodic with respect to $\mu_2$;*
3. *$\mu_2$ is recurrent for $Q$;*
4. *$Q$ admits no $\sigma$-finite invariant measure absolutely continuous with respect to $\mu_2$.*

*Proof:* It is clear that $Q$ is a nonsingular, 4-to-1 continuous endomorphism on $\mathbf{T}^2$. (It is enough to show there exists a partition $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ of $\mathbf{T}^2$ into 4 sets such that the restriction of $Q$ to each set is one-to-one and onto with respect to $\mu_2$. The partition into four sets with endpoint coordinates $0, 1$, and $\frac{1}{2}$ obviously works.) Since $S$ is exact with respect to $m$, and since $S$ is ergodic with respect to $\mu$, it follows by [1] that $Q$ is ergodic with respect to $\mu_2$.

Let $\omega_m$ denote the Radon–Nikodým derivative of $S$ with respect to $m$. Then clearly $\omega_m \equiv 1$. Let $\omega_\mu$ denote the Radon–Nikodým derivative of $S$ with respect to Hamachi measure. If $\omega_{m_2} \equiv \omega_2$ denotes the Radon–Nikodým derivative of $Q$ with respect to $m_2$, since $\omega_\mu$ is $Q^{-1}(\mathcal{B}\times\mathcal{B})$-measurable, it follows that $\omega_2(x,y) = \omega_\mu(y)$ for every $(x,y)$. The fact that $Q$ is type *III* for $m_2$ now follows exactly as in [12]. ∎

The following corollary can be proven using the same methods as in the circle case.

COROLLARY 7.1: *On the d-dimensional torus, there exist maps $f$ with one of the following sets of properties:*

1. *$f$ is a toral group endomorphism and $f$ type III with respect to some ergodic and conservative Borel measure.*
2. *$f$ is type III ergodic and conservative with respect to Lebesgue measure, and $f$ is $C^1$ expanding.*

Regarding $Q$ as a map of $\mathbf{R}^2/\mathbf{Z}^2$, we note that $Q(-(x,y)) = -Q(x,y)$ for all $(x,y) \in \mathbf{T}^2$. We now review some basic facts about the Weierstraß elliptic $\wp$ function in conjunction with a classical method of constructing analytic maps of the sphere given by Lattès [22].

7.2 LATTÈS EXAMPLES ON THE SPHERE. We consider the Weierstraß elliptic function of the complex plane $\mathbf{C}$; i.e., a meromorphic function on $\mathbf{C}$ which is periodic with respect to a given lattice and is even. In our case we are primarily interested in the lattice $L = \{m + in \colon m,n \in \mathbf{Z}\}$. We recall that

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in L,\, w\neq 0}\left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right]$$

satisfies the definition of an even elliptic function and can be regarded as a map from the period parallelogram $\mathbf{C}/L$ which is homeomorphic to $\mathbf{T}^2$. Furthermore $\wp\colon \mathbf{T}^2 \to \mathbf{S}^2 \cong \mathbf{T}^2/z \sim -z$ is a two-fold branched covering of the sphere by the torus.

Using the identification $z = x + iy = (x,y) \in \mathbf{C}$, when no confusion arises, $Q$ defines a complex endomorphism on $\mathbf{T}^2$ such that $Q(-z) = -Q(z)$. We can pass to the quotient space to obtain an analytic (rational) map of the sphere such that $\wp \circ Q = R \circ \wp$. In fact, using a classical "angle doubling" formula for $\wp$, and the fact that $Q(z) = 2z \,(\mathrm{mod}\, L)$, we obtain the following explicit formula for $R$ in this case:

$$R(z) = \frac{(z^2+1)^2}{4z(z^2-1)}.$$

If we put Lebesgue measure $m_2$ on the torus, then $Q$, and hence $R$ (using the obvious factor measure) will be isomorphic to the one-sided Bernoulli shift of entropy $\log 4$. By varying the lattice and the integer in the original endomorphism on $\mathbf{C}$, we obtain the earliest examples known of rational maps of the sphere with Julia set the whole sphere.

By varying the measure on $\mathbf{T}^2$, we obtain different factor measures on $\mathbf{C}_\infty$ in the obvious way; i.e., by using the measure $\alpha \circ \wp^{-1}$ on the sphere if $\alpha$ is the measure on the torus. The map $\wp$ gives a 2-set partition (not unique) of $\mathbf{T}^2$ minus exactly 4 branch points with the property that the restriction of $\wp$ to each atom of the partition maps injectively onto the sphere minus 4 points. We will fix from now on a choice of partition and call the 2 disjoint sets $A_0$ and $A_1$; they can be chosen to be connected, but we will not use that here. We choose $A_0$ to be the union of 2 atoms from the partition $\mathcal{P}$ defined earlier for $Q$. We define $\wp_0 = \wp|_{A_0}$ and $\wp_1 = \wp|_{A_1}$. Using this partition, any nonatomic Borel $\sigma$-finite measure $\rho$ on $\mathbf{C}_\infty$ gives rise to a measure on the torus via the map $\wp$ as follows. For each set $C \in \mathcal{B}_2$, define $-C = \{(-x,-y) \,(\mathrm{mod}\, L)\colon (x,y) \in C\}$. Recall that $Q(-C) = -Q(C)$ and $Q^{-1}(-C) = -[Q^{-1}(C)]$.

If $C = \wp^{-1} \circ \wp(C)$, we call $C$ a saturated set (under $\wp$). We can form the saturation of any set $B \in \mathcal{B}_2$ by $B_* \equiv \wp^{-1} \circ \wp(B) \supseteq B$. It is clear that $B_* = B \cup -B$, where the union may or may not be disjoint.

LEMMA 7.1: *Let $\rho$ be any nonatomic, nonsingular Borel measure for $R$. Then there exists an associated lifted measure $\rho_2$ on $\mathbf{T}^2$ such that:*

1. *$\rho_2$ is nonsingular for $Q$;*
2. *if $\rho$ is ergodic for $R$, then $\rho_2$ has at most 2 ergodic components with respect to $Q$;*
3. *if $\rho$ is ($\sigma$-)finite, then so is $\rho_2$;*
4. *if $\rho$ is invariant for $R$, then $\rho_2$ is invariant for $Q$.*

*Proof:* Given any $C \in \mathcal{B}_2$, we can write $C = C_0 \cup C_1 \cup C_{br}$ , where $C_0 = C \cap A_0$, $C_1 = C \cap A_1$, and $C_{br} = C \cap$(branch points of $\wp$). Clearly the union is disjoint, and any of these sets in the union could be empty. Define

$$\begin{aligned}\rho_2(C) &= \tfrac{1}{2}\rho(\wp_0(C_0)) + \tfrac{1}{2}\rho(\wp_1(C_1)) \\ &= \tfrac{1}{2}\rho(\wp_1(-C_0)) + \tfrac{1}{2}\rho(\wp_1(C_1)) \\ &= \tfrac{1}{2}\rho(\wp_0(C_0)) + \tfrac{1}{2}\rho(\wp_0(-C_1)),\end{aligned}$$

since $\wp_i(C_i) = \wp_j(-C_i)$, $i,j = 1,2$, $i \neq j$. We remark that if $C$ is a saturated set, then $\rho_2(C) = \rho(\wp C)$, and for any measurable $B$, $\rho_2(B_*) = 0$ if and only if $\rho_2(B) = 0$.

To show 1, we suppose $\rho_2(C) = 0$. By the above formulas, this implies that $\rho_2(\wp^{-1}\wp(C)) = \rho_2(C_*) = \rho(\wp C_*) = 0$. By nonsingularity, $\rho(R^{-1}\wp(C_*)) = 0$, and it is easy to see that for any saturated sets $C_* \in \mathcal{B}_2$, $\wp \circ Q^{-1}C_* = R^{-1} \circ \wp C_*$, so it follows that $\rho_2(Q^{-1}C_*) = 0 = \rho_2(Q^{-1}C)$. Similarly we can show that if $\rho_2(Q^{-1}C) = 0$, $\rho_2(C) = 0$ as well, by saturating $Q^{-1}C$.

We will now prove 2. We assume that $\rho$ is ergodic for $R$ and let $B$ be a positive measure $Q$-invariant set in $\mathcal{B}_2$. If $B$ is saturated, then by the ergodicity of $\rho$ for $R$, we have that $B$ has full $\rho_2$ measure. Furthermore, we can show that $B \cap -B$, which is saturated, is also $Q$ invariant. Therefore it has full or zero measure. Suppose then that $\rho_2(B \cap -B) = 0$ (or we are done because if not, then $B$ is saturated hence has full measure). Then since the set $B \cup -B$ is invariant and saturated, we now have 2 disjoint sets, $B$ and $-B$, each of positive measure, disjoint, and invariant. Their union has full measure; this follows from the ergodicity of $R$ with respect to $\rho$ and the fact that $B \cup -B$ is saturated. Then from the discussion it follows that $\wp$ maps $B$ injectively onto $\mathbf{C}_\infty$. From this it follows that no smaller set can be invariant; i.e., the ergodic decomposition can have at most 2 atoms in it, each of which completely covers the sphere. Therefore there are at most two distinct ergodic components.

Part 3 of Lemma 7.1 follows easily. Finally we establish the invariance of $\rho_2$ for $Q$ when $R$ preserves $\rho$. Let $\mathcal{A}$ denote the $\sigma$-algebra of Borel sets on the sphere. If $C$ is a saturated set in $\mathcal{B}_2$, then $\rho_2(Q^{-1}C) = \rho_2(C)$ by the hypothesis on $R$ and the above discussion. So it is enough to check invariance for invariance for $C \subset A_0$ (or $A_1$); assume $C = C \cap A_0 \in \mathcal{A}$. Then $C \cap (-C) = \emptyset$ and $(Q^{-1}C) \cap Q^{-1}(-C) = (Q^{-1}C) \cap -(Q^{-1}C) = \emptyset$. Then

$$\rho_2(C) = \tfrac{1}{2}\rho(\wp_0 C) = \tfrac{1}{2}\rho(\wp C) = \tfrac{1}{2}\rho(R^{-1}(\wp_0 C)) = \tfrac{1}{2}\rho(\wp(Q^{-1}C)).$$

Writing $Q^{-1}C = (Q^{-1}C)_0 \cup (Q^{-1}C)_1$, this equals

$$\tfrac{1}{2}[\rho(\wp_0(Q^{-1}C)_0) + \rho(\wp_1(Q^{-1}C)_1)] = \rho_2(Q^{-1}C).$$

This concludes the proof. ∎

7.3 Type *III* rational maps of the sphere. We construct the type *III* map on $\mathbf{T}^2$ as above, using the measure $\mu_2$, which is the product of one-dimensional Lebesgue measure $m$ with Hamachi measure $\mu$. On the sphere we use the natural factor measure defined by the factor map $\wp$; i.e., define $\nu(A) = \mu_2(\wp^{-1}(A))$ for every Borel set $A$ on $\mathbf{S}^2 \cong \mathbf{C}_\infty$. It is clear that $\nu$ is ergodic and conservative for the factor map $R$ defined above, so the following holds.

THEOREM 7.2: *There exists a Borel measure $\nu$ on $\mathbf{C}_\infty$ such that, with respect to the rational map $R(z) = (z^2+1)^2/4z(z^2-1)$:*

1. *$\nu$ is supported on the Julia set of $R$ (which is the whole sphere);*
2. *$R$ is a 4-to-1 map with respect to $\nu$;*
3. *$R$ is continuous, nonsingular, conservative, and ergodic with respect to $\nu$;*
4. *$R$ admits no $\sigma$-finite invariant measure absolutely continuous with respect to $\nu$.*

*Proof:* From the discussion above and Lemma 7.1, it is clear that we only need to check 4. We suppose that $R$ admits an invariant measure $\rho \sim \nu$, and that $\rho$ is $\sigma$-finite. Then $\rho$ is ergodic since $\nu$ is, so we lift $\rho$ to an invariant measure for $Q$ on $\mathbf{T}^2$ as in the preceding lemma. We denote by $\rho_2$ the lifted measure on $\mathbf{T}^2$, and it has all the properties listed in Lemma 7.1. It remains to show that $\rho_2$ is equivalent to $\mu_2$ which would contradict the hypothesis on $\mu_2$.

One can easily establish that $\mu_2 \ll \rho_2$ since:

$$\begin{aligned}\rho_2(C) = 0 &\Rightarrow \rho(\wp_0(C_0)) + \rho(\wp_1(C_1)) + \rho(\wp_0(-C_1)) + \rho(\wp_0(-C_1)) = 0\\ &\Rightarrow \rho(\wp C) = 0\\ &\Rightarrow \nu(\wp_0(C_0)) + \nu(\wp_1(C_1)) + \nu(\wp_0(-C_1)) + \nu(\wp_0(-C_1)) = 0\\ &\Rightarrow \mu_2(C) = 0.\end{aligned}$$

If the measure $\rho_2$ is ergodic for $Q$, then we suppose there exists a measurable set $C$ such that $\mu_2(C) = 0$ and $\rho_2(C) > 0$. We can generate an invariant set for $Q$ by $C$ which then has full $\rho_2$ measure by ergodicity of $\rho_2$; its complement will have $\rho_2$-, hence $\mu_2$-measure 0. It cannot happen that $\mu_2$ gives measure 0 to a set and its complement. Therefore we assume, using Lemma 7.1, that $\rho_2$ has two ergodic components with respect to $Q$; then we will write the measure as the sum of two ergodics: $\rho_2 = \frac{1}{2}\rho_2^0 + \frac{1}{2}\rho_2^1$. We repeat the above argument on each component separately; i.e., any set $C$ such that $\mu_2(C) = 0$ but $\rho_2(C) > 0$ must lie completely in one ergodic component of $\rho_2$ (otherwise we repeat the above argument verbatim). Therefore $\mu_2 \sim \rho_2^0$ say. The set $C$ generates an invariant set of full $\rho_2^1$ measure whose complement has measure 0. Hence $\mu_2$ gives both $C$ and its complement measure 0 and the contradiction establishes the result. ∎

*Remarks:* 1. The Lattès examples are constructed for any endomorphism of the form $Qz = nz$ for any $n \geq 2$. In this way we can construct type *III* examples of degree $n^2$.

2. We can also vary the lattice used in the definition of the Weierstraß elliptic function to obtain different conformal equivalence classes of maps. The measure

theoretic properties will remain the same however, as changing the lattice is a measure theoretic isomorphism.

3. Because of averaging that occurs in the Weierstraß factor map, our method does not immediately lead to a $C^1$ type *III* version of the example.

## References


[1] J. Aaronson, M. Lin and B. Weiss, *Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products*, Israel Journal of Mathematics **33** (1979), 198–224.

[2] H. Bruin, *Induced maps, Markov extensions and invariant measures in one-dimensional dynamics*, Communications in Mathematical Physics **168** (1995), 571–580.

[3] B. Derrida, A. Gervois and Y. Pomeau, *Iteration of endomorphisms on the real axis and representation to numbers*, Annales de l'Institut Henri Poincaré, Ser. A **29** (1978), 305–356.

[4] K. Dajani and J. Hawkins, *Examples of natural extensions of nonsingular endomorphisms*, Proceedings of the American Mathematical Society **120** (1994), 1211–1217.

[5] S. Eigen and C. Silva, *A structure theorem for $n$-to-1 endopmorphisms and existence of non-recurrent measures*, Jurnal of the London Mathematical Society **40** (1989), 441–451.

[6] P. Góra, *Properties of invariant measures for piecewise expanding one-dimensional transformations*, Ergodic Theory and Dynamical Systems **14** (1994), 475–492.

[7] P. Góra and B. Schmitt, *Un exemple de transformation dilatante et $C^1$ par morceaux de l'intervalle, sans probabilité absolument continue invariante*, Ergodic Theory and Dynamical Systems **9** (1989), 101–113.

[8] G. Gyögyi and P. Szépfalusy, *Properties of fully developed chaos in one-dimensional maps*, Journal of Statistical Physics **34** (1984), 451–475.

[9] T. Hamachi, *On a Bernoulli shift with non-identical factor measures*, Ergodic Theory and Dynamical Systems **1** (1981), 273–284.

[10] T. Hamachi and M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorems*, Seminar on Mathematical Sciences. Keio Univeristy, 1981.

[11] J. Hawkins and C. Silva, *Remarks on recurrence and orbit equivalence of non-singular endomorphisms*, Dynamic Systems Proceedings, University of Maryland, Lecture Notes in Mathematics **1342**, Springer-Verlag, Berlin, 1988, pp. 281–290.

[12] J. Hawkins and C. Silva, *Non invertible transformations admitting no absolutely continuous $\sigma$-finite invariant measures*, Proceedings of the American Mathematical Society **111** (1991), 455–463.

[13] M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **49** (1979), 5–233.

[14] F. Hofbauer and G. Keller, *Quadratic maps without asymptotic measure*, Communications in Mathematical Physics **127** (1990), 319–337.

[15] S. Kakutani, *On equivalence of infinite product measures*, Annals of Mathematics **49** (1948), 214–224.

[16] Y. Katznelson, *Sigma-finite invariant measures for smooth maps of the circle*, Journal d'Analyse Mathématique **31** (1977), 1–18.

[17] Y. Katznelson, *The action of diffeomorphism of the circle om the Lebesgue measure*, Journal d'Analyse Mathématique **36** (1979), 156–166.

[18] G. Keller, *Exponents, attractors and Hopf decompositions for interval maps*, Ergodic Theory and Dynamical Systems **10** (1990), 717–744.

[19] G. Keller and T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps*, Communications in Mathematical Physics **149** (1992), 31–69.

[20] K. Krzyzewski and W. Szlenk, *On invariant measures of expanding differentiable mappings*, Studia Mathematica **33** (1969), 83–92.

[21] A. Lasota and J. Yorke, *On existence of invariant measures for piecewise monotonic transformations*, Transactions of the American Mathematical Society **186** (1973), 481–488.

[22] S. Lattès, *Sur l'iteration des substitutions rationelles*, Comptes Rendus de l'Académie des Sciences, Paris **166** (1919), 26–28.

[23] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer, Berlin, 1987.

[24] D. Ornstein, *On invariant measures*, Bulletin of the American Mathematical Society **66** (1960), 297–300.

[25] A. Quas, *Non-ergodicity for $C^1$ expanding maps and $g$-measures*, Ergodic Theory and Dynamical Systems **16** (1996), 531–542.

[26] V. Rohlin, *On the fundamental ideas of measure theory*, American Mathematical Society Translations **71** (1952), 1–54.

[27] M. Rychlik, *Bounded variation and invariant measures*, Studia Mathematica **76** (1983), 69–80.

[28] C. Silva and P. Thieullen, *The sub-additive ergodic theorem and recurrence properties of Markovian transformations*, Journal of Mathematical Analysis and Applications **154** (1991), 83–99.